

SOLUTIONS 11-20 less 12

11. Billy has 100 dimes, 100 nickels and 100 pennies. He must use exactly 90 of these coins to buy a \$4.00 toy. The number of different ways that this can be done is

- a) 8 b) 9 c) 11 d) 10 e) none of these

Solution

If D , N , and P are the number of dimes, nickels and pennies respectively, then

$$10D + 5N + P = 400$$

$$D + N + P = 90$$

$$9D + 4N = 310$$

$$D = \frac{310 - 4N}{9}$$

The last equation, with $0 \leq N \leq 90$ and $0 \leq D \leq 90$, gives integer values of D if and only if $N = 1 + 9k$, $k = 0, 1, 2, \dots, 8$. Thus there are exactly 9 ways to choose the payment.

For example if $k = 3$ then $N = 28$, $D = 22$, $P = 40$.

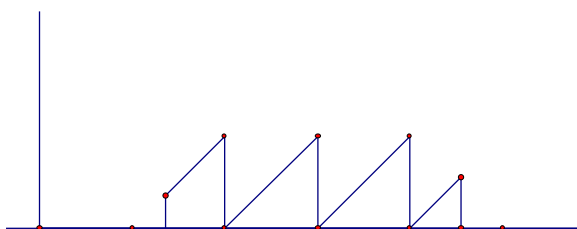
13. The fractional part of a positive number x is denoted by $\{x\}$. For example $\{2\frac{1}{3}\} = \frac{1}{3}$ and $\{3.17\} = 0.17$. If R is the region in the (x,y) -plane above the x -axis, below $y = \{x\}$, and between $x = 1\frac{1}{3}$ and $x = 4\frac{1}{2}$, then the area of R is

- a) $57/36$ b) $111/72$ c) $113/72$ d) $55/36$ e) none of these

Solution

Three 1 by 1 triangles less one $1/3$ by $1/3$ triangle plus one $1/2$ by $1/2$ triangle

$$\text{Area} = 3(1/2) - 1/18 + 1/8 = 113/72$$



14. Suppose that 25% of all the wise people are nice and half of all the nice people are wise. If 25% of the people are neither nice nor wise, then what percent of the people are both nice and wise?

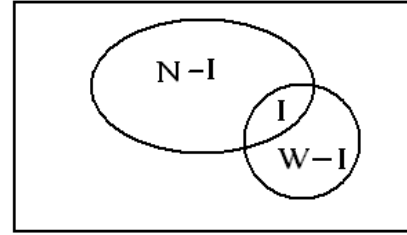
- a) 10 b) 20 c) 20 d) 15 e) none of these

SOLUTION

Let N be the number of nice, W the number of wise, I the number in their intersection and P the total number of people. Then we have

$$I = W/4, I = N/2,$$

$$\text{neither nice nor wise} = P - W - N + I = P/4.$$



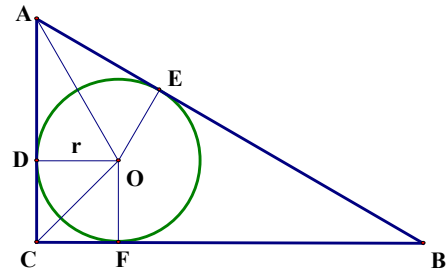
Thus $P - 4I - 2I + I = P/4$, and $I = 3P/20 = .15P$

15. If a circle is inscribed in a triangle with vertex angles 30° , 60° , and 90° , then the area of the triangle divided by the area of the circle is

- a) $(3+2\sqrt{3})\pi$ b) $(3+3\sqrt{3})\pi$ c) $(2+3\sqrt{3})\pi$ d) $(1+2\sqrt{3})\pi$ e) none of these

SOLUTION

Let r be the circle radius and A, B, C be the triangle vertices with angle measures 60° , 30° , 90° respectively. Since a tangent to a circle is perpendicular to the radius, then AOD and AOE are congruent $30, 60, 90$ right triangles. Thus $AD = r\sqrt{3}$. Also COD and COF are congruent $45, 45, 90$ right triangles. Thus $DC = r$. Hence the area of triangle $ABC =$

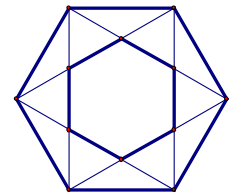


$$(AC)(CB)/2 = (AC)(AC\sqrt{3})/2 = (AD + DC)^2 \sqrt{3}/2 = r^2(1 + \sqrt{3})^2 \sqrt{3}/2 = \sqrt{3} r^2(2 + \sqrt{3})$$

Since the circle area is πr^2 , then the triangle area divided by the circle area is $(3 + 2\sqrt{3})/\pi$.

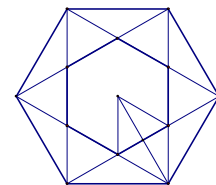
Solution 2. This can also be solved using the in-circle result:
triangle area = (circle radius) times (triangle semi-perimeter)

16. The small regular hexagon GHIJKL is constructed from the large regular hexagon ABCDEF as shown in the figure. If the length of each of the sides of ABCDEF is one unit, then the area of the smaller hexagon is



- a) $\sqrt{3}/2$ b) $\sqrt{3}/3$ c) $\sqrt{2}/2$
d) $2\sqrt{2}/3$ e) none of these

Solution : Let M be the center of the hexagons. Triangles MBC and MGH are both equilateral. $MB = 1$, $MN = 1/2$ and $NH = 1/(2\sqrt{3})$, thus area of triangle MGH = $1/(4\sqrt{3})$. Hence area (GHIJKL) = $6/(4\sqrt{3}) = \sqrt{3}/2$.



17. Let C be the units digit, B the tens digit and A the hundreds digit. For example if $A = 4$, $B = 2$ and $C = 3$, then ABC represents the integer four hundred and twenty three and BC represents twenty three. Note that leading digits cannot be 0. For the only triple (A,B,C) such that $(BC)^2 = ABC$, the sum of the digits, $A + B + C$, is

- a) 12 b) 13 c) 14 d) 15 e) none of these

Solution: Let $L = (10B + C)^2 = 100B^2 + 20BC + C^2$, and $R = 100A + 10B + C$. We seek triples (A,B,C) such that $L = R$.

The units digit of L is equal to the units digit of C^2 , and the units digit of R is equal to C . Thus C must be 0, 1, 5 or 6. Also $3 \leq B \leq 1$, since L is a three digit integer. Thus the only possible (B,C) pairs are $(1,0), (1,1), (1,5), (1,6), (2,0), (2,1), (2,5), (2,6), (3,0), (3,1), (3,5)$ or $(3,6)$. Only the pair $(B,C) = (2,5)$ can satisfy the given equality and thus $A = 6$. Hence $(6,2,5)$ is the only triple satisfying the required equality.

18. The wealthy mother died leaving an estate of G gold coins, where G is an integer greater than 30. She has D daughters and they inherit the coins as follows. The eldest receives 1 coin plus $1/k$ of those remaining, where k is an integer. The second eldest receives 2 coins plus $1/k$ of those remaining. The third eldest receives 3 coins plus $1/k$ of those remaining. This pattern continues for the rest of the daughters, except that the youngest receives all the coins that were not inherited by her sisters. If each of the daughters received the same number of coins, then $D + G$ is

- a) $6k$ b) $k^2 + k$ c) $k^2 + 1$ d) $(k+1)^2$ e) none of these

Solution : This problem, for $k = 7$, using father and sons, was posed and solved by the famous mathematician Fibonacci in the thirteenth century.

Let S_1 be the number of coins received by the eldest, S_2 the number of coins received by the second eldest, and R the number of coins remaining after the eldest has been given her share. Then

$$S_1 = 1 + \frac{G-1}{k}, \quad R = G - S_1 = G - \left(1 + \frac{G-1}{k}\right) = \frac{(G-1)(k-1)}{k} \quad S_2 = 2 + \frac{R-2}{k}.$$

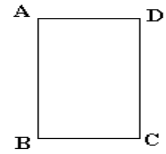
Since each share is the same, then $S_1 = S_2$, and solving gives

$$G = k^2 - 2k + 1 \text{ and thus } S_1 = k - 1.$$

Since each of the D daughters gets S_1 coins, then $D = G/S_1 = k - 1$. Thus

$$D + G = k^2 - k.$$

19. Given the rectangle $ABCD$ with $AB = 6$ inches and $AD = 3$ inches. The rectangle is folded so that vertices A and C coincide. What is the length of the crease?



- a) $\frac{3}{2}\sqrt{6}$ b) $\frac{2}{3}\sqrt{5}$ c) $\frac{2}{3}\sqrt{6}$ d) $\frac{3}{2}\sqrt{5}$ e) none of these.

SOLUTION

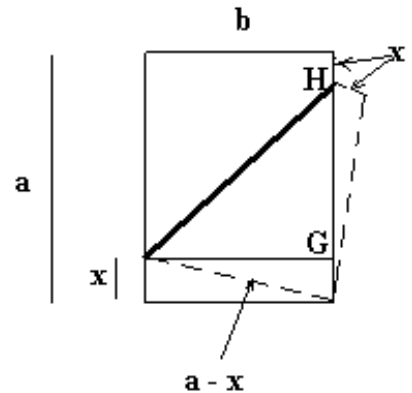
Let a and b be the dimensions of the rectangle. The heavy line is the fold and the dashed lines are the edges after the fold. The small right triangle gives

$$x^2 + b^2 = (a - x)^2 \Rightarrow 2x = \frac{a^2 - b^2}{a} \Rightarrow a - 2x = \frac{b^2}{a}.$$

The small right triangles are congruent, thus $GH = a - 2x$

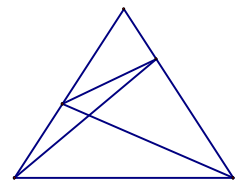
$$\text{Crease} = \sqrt{b^2 + (a - 2x)^2} = \frac{b}{a} \sqrt{a^2 + b^2}$$

For $a = 6$ and $b = 3$, we have $\text{Crease} = \frac{3}{2}\sqrt{5}$.



20. Triangle ABC is isosceles ($AB = AC$). Point D is on side AC , such that $\angle DBC = 60^\circ$. Point E is on side AB , such that $\angle ECB = 50^\circ$. The triangle is shown in the figure but not to scale. If $\angle BAC = 20^\circ$ then $\angle AED$, in degrees, is

- a) 60 b) 45 c) 50 d) 40 e) none of these



Solution

Construct FD parallel to BC , with F on AB . Draw CF intersecting BD at G . Note that triangles BGC and FGD are equilateral. Since $\angle CBE = 80^\circ$, then $\angle BEC = 180 - 80 - 50 = 50 = \angle BCE$. Thus $BC = BE = BG$ and triangle BEG is isosceles, with vertex angle 20 and base angles 80 .

Hence $\angle DFE = \angle FEG = \angle EGD = 100$, since they are all supplements of 80 . Thus $\angle EGF = \angle EFG = 100 - 60 = 40$, and $EF = EG$. Finally

$\angle AED = \angle FED = \frac{\angle FEG}{2} = 50$, since triangles EDF and EDG are congruent (SSA).

