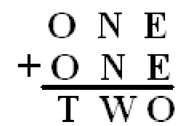
Solutions 2007 prob 11-20

11. How many integers x are there such that $1 \le x \le 100$ and $x^3 + 3x + 1$ is divisible by 5? Note that a is divisible by b means that if a is divided by b, then the remainder is 0.

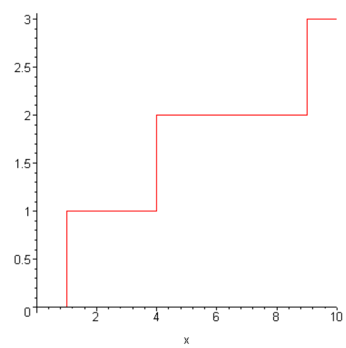
SOLUTION The units digit of $x^3 + 3x + 1$ must be either 0 or 5 and thus the units digit of x must be 1,2,6 or 7. Thus there are 40 x's satisfying the requirements.

12. In the cryptogram the letters represent distinct digits. There are no carries in the addition, thus **O, N**, and **E** represent digits that are all less than 5 and equal to or greater than 0. How many different addition problems could this cryptogram represent?



SOLUTION First note that none of the letters can represent zero. The units column implies that O is 2 or 4. If O = 2, then E = 1 and T = 4, and thus N = 3. If O = 4, then E = 2 and T = 8, and thus N is again 3. Hence there are two possibles.

13. Find the area in the first quadrant bounded by the line $x = 1$ on the left, the line $x = 10$ on the right, the line $y = 0$ below, and the function $y = [[x]^{1/2}]$ above. The bracket function is used twice and denotes the greatest integer function, for example $[3.72] = 3$. Also $a^{1/2} = \sqrt{a}$.

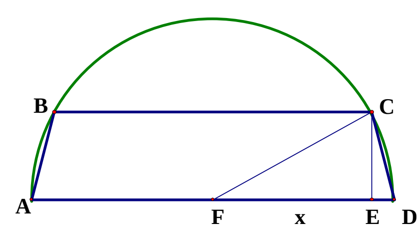


SOLUTION The graph is shown and thus the area is 16

14. The absolute value of x is denoted by $|x|$. The product of all real values of x satisfying the equation $2\,|x|^2 + |x| = 6$ is

SOLUTION If x is real and greater than 0 then $|x| = x$ and $2x^2 + x = 6 \Rightarrow x = 3/2$. If x is real and less than 0 then $|x| = -x$ and $2x^2 - x = 6 \Rightarrow x = -3/2$. Hence the two possible values for x are 3/2, or -3/2, and their product is -9/4.

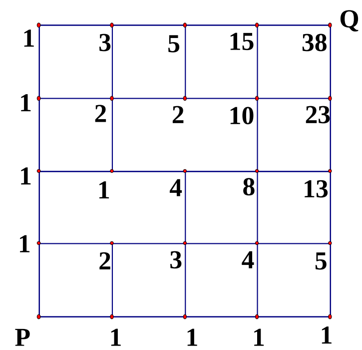
15. A trapezoid $ABCD$ is inscribed in a semicircle of radius 2. They share the same base AD. If the lengths of segments AB and DC are both 1, then the area of $ABCD$ is



SOLUTION Let F be the circle center and CE be perpendicular to AD. If x is the length of segment FE, then 2-x is the length of segment ED. Segment CE is the altitude of two right triangles and thus $1^2-(2-x)^2=2^2-x^2 \;\Rightarrow\; x=7/4 \Rightarrow CE=\sqrt{15}/4$.

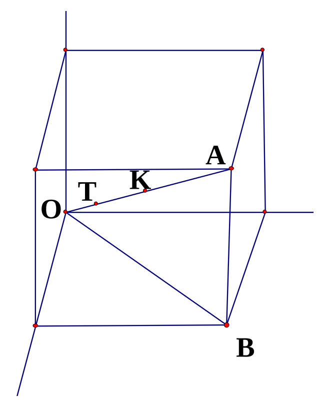
Hence the area $ABCD = \dfrac{(4+7/2)}{2}\dfrac{\sqrt{15}}{4}=\dfrac{15\sqrt{15}}{16}$.

16. How many distinct paths are there from P to Q if you must stay on the line segments and at each intersection you must go up or to the right?



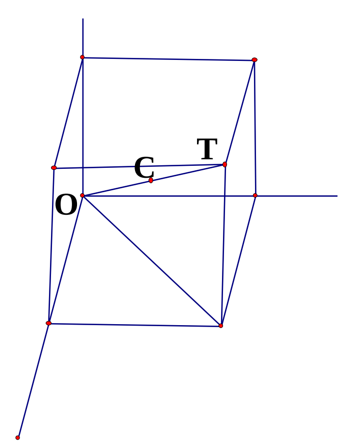
SOLUTION The numbers at each node indicate the number of paths from below or from the left that feed that node.

17. Given a sphere of radius 1 inscribed in a cube (tangent to all 6 faces). A smaller sphere is nestled in the corner such that it is tangent to the given sphere and also tangent to three intersecting faces of the cube. The radius of the smaller sphere, to two decimals, is



SOLUTION Note that the diagonal of a cube with side length x is $x\sqrt{3}$. The top figure is the cube circumscribing the large sphere with side length = 2, thus $OA = 2\sqrt{3}$. K is the center of the large sphere and T is the point of tangency of the large sphere and the small sphere. The lower figure is an expanded drawing of the cube circumscribing the small sphere of radius r, C is the center of the small sphere. Thus $KO = KT + TC + CO$.

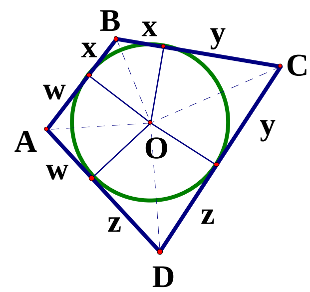
We have KT = the radius of the large sphere = 1. $TC = r$, the radius of the small sphere. $CO = r\sqrt{3}$, half the diagonal if the smaller cube with side length 2r. $KO=\sqrt{3}$, half the diagonal of the large cube. Thus



$\sqrt{3}=1+r+r\sqrt{3}$ and $r=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}\cong 0.27$

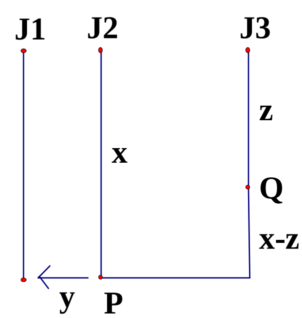
18. A circle is inscribed in the quadrilateral *ABCD* as shown (not to scale). The sides *AB*, *BC*, and *CD* have lengths 8, 5, and 10 respectively. The length of side *DA* is

SOLUTION Let *O* be the circle center and *w*,*x*,*y*,*z* be the respective distances from the vertices *A*,*B*,*CD* to the tangent points. The two distances of a pair, from a given vertex, are equal, by congruent triangles. Thus $x+w=8$, $x+y=5$, $y+z=10$. Adding the first and third eqns. gives $x+w+y+z=18$ Thus $w+z=13$, since $x+y=5$.



19. Three Jeeps, J_1, J_2, J_3, are at base camp on the edge of a desert with full tanks of gas. They are to help J_1 drive to an oasis deep into the desert with J_2 and J_3 returning safely back to base camp. J_1 and J_2 travel together for a spell and then J_2 refills J_1. J_1 then continues to the oasis and J_2 turns around to head back towards base until empty where it is met by J_3. J_3 has stayed at base camp, to later head out and share gas with the returning J_2, so that they can both get back to base. If the tanks hold 30 gallons and the Jeeps get 20 miles per gallon, then the maximum distance, in miles, that the oasis can be from base camp is

SOLUTION The figure shows the paths of the three Jeeps. For clarity they are shown parallel but in fact they are together. It is convenient to measure gas in miles rather than gallons. J2 travels with J1 to a point *P*, shares *y* miles of gas with J1, and then returns to a point *Q* where J3 helps J2 back to camp. J1 continues to the oasis and thus penetrates 600 + *y* from base camp. Thus we have the following requirements:

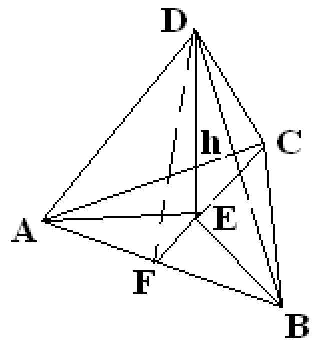


(a) maximize y
(b) At *P*, J1 cannot accept more than *x* miles of gas else overflow. Thus $y \le x$.
(c) J2 arrives empty at *Q* else it could have gone farther before sharing with J1.
(d) J2 can come out no more than 200 miles and be able to bring self and J2 back to base. Thus $z \le 200$.

From (c): $600 = x + y + x - z \Rightarrow y = 600 + z - 2x$. Combining with (b) and (d) gives $y \le 600 + 200 - 2y \Rightarrow y \le 800/3$. Thus max dist to oasis = 600+800/3

20. Let *ABCD* be a regular tetrahedron with each of its six sides of length 1. Let *E* be any point in face *ABC*. Let *s* the sum of the distances from *E* to the faces *DAB*, *DBC* and *DCA*, and *S* be the sum of distances from *E* to the edges *AB*, *BC* and *CA*. Find S/s .

SOLUTION Let *E* be equidistant from vertices *A*, *B*, and *C*, then *ED* is the height *h* of the tetrahedron *ABCD*. Let *EF* be the altitude of triangle ABE to side AB. Note that *ED* is a side in the right triangle *FED* , *FE* is a side of the 30-60-90 triangle *BFE* with *FB*=1/2, and *FD* is the hypotenuse of the 30-60-90 triangle *AFD* with *AF*=1/2. Combining and using the Pythagorean theorem, gives $h = \sqrt{2}/\sqrt{3}$.



Now let *E* be *any* point in the face *ABC*. Let x, y, z be the distances from *E* to the faces *DAB*, *DBC*, *DCA*, respectively. Let u, v, w be the distances from *E* to the edges *AB*, *BC*, *CA*, respectively. The volume of a tetrahedron is 1/3 the area of the base times the height and we calculate *V(ABCD)* in two ways, where *V* denotes volume. First method, add the volumes of the three tetrahedrons with bases *AEB*, *BEC*, and *CEA*.

$$V(ABCD) = \frac{1}{3}h(1)(\frac{u}{2} + \frac{v}{2} + \frac{w}{2}) = \frac{S\sqrt{2}}{6\sqrt{3}}.$$

The second way to find *V(ABCD)* is to sum the volumes of three tetrahedrons with bases *DAB*, *DBC*, and *DCA* and altitudes *x*, *y*, *z* respectively. Each base area is $\sqrt{3}/4$ and

$$V(ABCD) = \frac{1}{3}\frac{\sqrt{3}}{4}(x + y + z) = \frac{s\sqrt{3}}{12}.$$

Thus $\dfrac{S}{s} = \dfrac{3}{2\sqrt{2}}$